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# MULTIPLIERS AND WIENER-HOPF OPERATORS ON WEIGHTED $L^p$ SPACES

VIOLETA PETKOVA

Abstract. We study the multipliers $M$ (bounded operators commuting with the translations) on weighted spaces $L^p_\omega(\mathbb{R})$. We establish the existence of a symbol $\mu_M$ for $M$ and some spectral results for the translations $S_t$ and the multipliers. We also study the operators $T$ on the weighted space $L^p_\omega(\mathbb{R}^+)$ commuting either with the right translations $S_t$, $t \in \mathbb{R}^+$, or left translations $P^+S_{-t}$, $t \in \mathbb{R}^+$, and we establish the existence of a symbol $\mu$ of $T$. We characterize completely the spectrum $\sigma(S_t)$ of the operator $S_t$ proving that

$$\sigma(S_t) = \{z \in \mathbb{C} : |z| \leq e^{t\alpha_0}\},$$

where $\alpha_0$ is the growth bound of $(S_t)_{t\geq 0}$. We obtain a similar result for the spectrum of $(P^+S_{-t})$, $t \geq 0$. Moreover, for an operator $T$ commuting with $S_t$, $t \geq 0$, we establish the inclusion $\overline{\mu(\mathcal{O})} \subset \sigma(T)$, where $\mathcal{O} = \{z \in \mathbb{C} : \operatorname{Im} z < \alpha_0\}$.

## 1. Introduction

Let $E$ be a Banach space of functions on $\mathbb{R}$. For $t \in \mathbb{R}$, define the translation by $t$ on $E$ by

$$S_t f(x) = f(x - t), a.e., \forall f \in E.$$

We call a multiplier on $E$, every bounded operator on $E$ commuting with $S_t$ for every $t \in \mathbb{R}$. For the multipliers on a Hilbert space we have the existence of a symbol and some spectral results concerning the translations and the multipliers are obtained by using this property of the multipliers (see [7], [8]). In the arguments exploited in [7], [8] the spectral mapping theorem of Gearhart [3] for semigroups in Hilbert spaces plays an essential role.

The first purpose of this paper is to extend the main results in [8], [7] concerning the existence of the symbol of a multiplier as well as the spectral results in the case where $E$ is a weighted $L^p_\omega(\mathbb{R})$ space. For general Banach spaces the characterization of the spectrum of the semigroup $V(t) = e^{tG}$ by the resolvent of its generator $G$ is much more complicated than for semigroups in Hilbert spaces (see for instance [4]). In particular, the statements of Lemma 1, 2 and 3 (see Section 2) are rather difficult to prove and for general Banach spaces this problem remains open. In this paper we restrict our attention to $L^p_\omega(\mathbb{R})$, $1 \leq p < \infty$, weighted spaces. The advantage that we take account is that the semigroup of the translations $(S_t)$ preserves the positive functions. For semigroups having this special property in the spaces $L^p_\omega(\mathbb{R})$ we have a spectral mapping theorem (see [1], [12], [13]). We obtain Theorems 1-4 for multipliers on $L^p_\omega(\mathbb{R})$ and in this work we explain only these parts of the proofs which are based on spectral mapping techniques and which are different from the arguments used to establish Theorems 1-4 in the particular case

$p = 2$ (see for more details [8], [7]).

For a Banach space $E$ denote by $E'$ the dual space of $E$. For $f \in E$, $g \in E'$, denote by $< f, g >$ the duality. Let $p \geq 1$, and let $\omega$ be a weight on $\mathbb{R}$. More precisely, $\omega$ is a positive, continuous function such that

$$\sup_{x\in\mathbb{R}} \frac{\omega(x+t)}{\omega(x)} < +\infty, \forall t \in \mathbb{R}.$$

Let $L^p_\omega(\mathbb{R})$ be the set of measurable functions on $\mathbb{R}$ such that

$$\|f\|_{p,\omega} = \Big(\int_{\mathbb{R}} |f(x)|^p \omega(x)^p dx\Big)^{1/p} < +\infty,\ 1 \leq p < +\infty.$$

Let $C_c(\mathbb{R})$ (resp. $C_c(\mathbb{R}^+)$ ) be the space of continuous functions on $\mathbb{R}$ (resp. $\mathbb{R}^+$) with compact support in $\mathbb{R}$ (resp. $\mathbb{R}^+$). Notice that $C_c(\mathbb{R})$ is dense in $L^p_\omega(\mathbb{R})$.
In the following we set $E = L^p_\omega(\mathbb{R})$ and we consider only Banach spaces having this form for $1 \leq p < +\infty$. In this case

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\bar{g}(x)\omega^2(x)dx$$

and

$$|\langle f, g\rangle| \leq \|f\|_{p,\omega}\|g\|_{q,\omega},\ \text{for } 1 < p < +\infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. For $p = 1$, we have

$$E' = L^\infty_\omega(\mathbb{R}) = \{f \text{ is measurable}:\ |f(x)|\omega(x) < \infty, a.e.\}$$

and

$$\|g\|_{\infty,\omega} = \text{esssup}\,\{|f(x)|\omega(x),\ x \in \mathbb{R}\}.$$

If $M$ is a multiplier on $E$ then, there exists a distribution $\mu$ such that

$$Mf = \mu * f,\ \forall f \in C_c(\mathbb{R}^+).$$

For $\phi \in C_c(\mathbb{R}^+)$, the operator

$$M_\phi : L^p_\omega(\mathbb{R}) \ni f \longrightarrow \phi * f$$

is a multiplier on $E$. Introduce

$$\alpha_0 = \lim_{t\to+\infty} \ln \|S_t\|^{\frac{1}{t}},\ \alpha_1 = \lim_{t\to+\infty} \ln \|S_{-t}\|^{\frac{1}{t}}.$$

It is easy to see that $\alpha_1 + \alpha_0 \geq 0$. Consider

$$U = \{z \in \mathbb{C},\ \operatorname{Im} z \in [-\alpha_1,\ \alpha_0]\}.$$

For an operator $T$ denote by $\rho(T)$ the spectral radius of $T$ and by $\sigma(T)$ the spectrum of $T$. It is well known that $\rho(S_t) = e^{\alpha_0 t}$, for $t \geq 0$.

Given a function $f$ and $a \in \mathbb{C}$, denote by $(f)_a$ the function

$$\mathbb{R} \ni x \longrightarrow f(x)e^{ax}$$

and denote by $\mathcal{M}$ the algebra of the multipliers on $E$. We note by $\hat{g}$ the Fourier transform of a function $g \in L^2(\mathbb{R})$. Our first result is a theorem saying that every multiplier on $E$ has a representation by a symbol.

**Theorem 1.** *Let $M$ be a multiplier on $E$. Then*
1) *For $a \in [-\alpha_1,\ \alpha_0]$, we have $(Mf)_a \in L^2(\mathbb{R})$, for every $f \in E$ such that $(f)_a \in L^2(\mathbb{R})$.*
2) *For $a \in [-\alpha_1,\ \alpha_0]$, there exists a function $\nu_a \in L^\infty(\mathbb{R})$ such that*

$$\widehat{(Mf)_a}(x) = \nu_a(x)\widehat{(f)_a}(x),\ \forall f \in E, \text{ with } (f)_a \in L^2(\mathbb{R}),\ a.e.$$

*Moreover, we have $\|\nu_a\|_\infty \leq C\|M\|$, $\forall a \in [-\alpha_1,\ \alpha_0]$.*
3) *If $\overset{\circ}{U} \neq \emptyset$, there exists a function $\nu \in \mathcal{H}^\infty(\overset{\circ}{U})$ such that*

$$\widehat{Mf}(z) = \nu(z)\hat{f}(z),\ z \in \overset{\circ}{U},\ \ \forall f \in C_c^\infty(\mathbb{R}),$$

*where $\widehat{Mf}(ia + x) = \widehat{(Mf)_a}(x)$, for $a \in ]-\alpha_1, \alpha_0[$, $f \in C_c^\infty(\mathbb{R})$.*

The function $\nu$ is called the **symbol** of $M$. The above result is similar to that established in [8], [7] and the novelty is that we treat Banach spaces $L^p_\omega(\mathbb{R})$ and not only Hilbert spaces. Define $\mathcal{A}$ as the closed Banach algebra generated by the operators $M_\phi$, for $\phi \in C_c(\mathbb{R})$. Notice that $\mathcal{A}$ is a commutative algebra. Our second result concerns the spectra of $S_t$ and $M \in \mathcal{M}$.

**Theorem 2.** *We have*

$$i)\ \sigma(S_t) = \{z \in \mathbb{C},\ e^{-\alpha_1 t} \leq |z| \leq e^{\alpha_0 t}\},\ \forall t \in \mathbb{R}. \tag{1.1}$$

*Let $M \in \mathcal{M}$ and let $\mu_M$ be the symbol of $M$.*
*ii) We have*

$$\overline{\mu_M(U)} \subset \sigma(M). \tag{1.2}$$

*iii) If $M \in \mathcal{A}$, then we have*

$$\overline{\mu_M(U)} = \sigma(M). \tag{1.3}$$

The equality (1.3) may be considered as a weak spectral mapping property (see [2]) for operators in the Banach algebra $\mathcal{A}$. On the other hand, it is important to note that if $M \in \mathcal{M}$, but $M \notin \mathcal{A}$, in general we have $\overline{\mu_M(U)} \neq \sigma(M)$. For the space $E = L^1(\mathbb{R})$, there exists a counter-example (see section 2 and [2]). Thus the inclusion in (1.2) could be strict.

In section 3, we obtain similar results for Wiener-Hopf operators on weighted $L^p_\omega(\mathbb{R}^+)$ spaces. In the analysis of Wiener-Hopf operators some new difficulties appear in comparison with the case of multipliers.

Let $\mathbf{E}$ be a Banach space of functions on $\mathbb{R}^+$. Let $p \geq 1$ and let $\omega$ be a weight on $\mathbb{R}^+$. It means that $\omega$ is a positive, continuous function such that

$$0 < \inf_{x \geq 0} \frac{\omega(x+t)}{\omega(x)} \leq \sup_{x \geq 0} \frac{\omega(x+t)}{\omega(x)} < +\infty, \forall t \in \mathbb{R}^+.$$

Let $L^p_\omega(\mathbb{R}^+)$ be the set of measurable functions on $\mathbb{R}^+$ such that

$$\int_0^\infty |f(x)|^p \omega(x)^p dx < +\infty.$$

Notice that $C_c(\mathbb{R}^+)$ is dense in $L^p_\omega(\mathbb{R}^+)$.

Let $P^+$ be the projection from $L^2(\mathbb{R}^-) \oplus L^p_\omega(\mathbb{R}^+)$ into $L^p_\omega(\mathbb{R}^+)$. From now we will denote by $\mathbf{S}_a$ the restriction of $S_a$ on $L^p_\omega(\mathbb{R}^+)$ for $a \geq 0$ and, for simplicity, $\mathbf{S}_1$ will be denoted by $\mathbf{S}$. Let $I$ be the identity operator on $L^p_\omega(\mathbb{R}^+)$.

**Definition 1.** *A bounded operator $T$ on $L^p_\omega(\mathbb{R}^+)$ is called a Wiener-Hopf operator if*

$$P^+\mathbf{S}_{-a}T\mathbf{S}_a f = Tf,\ \forall a \in \mathbb{R}^+,\ f \in L^p_\omega(\mathbb{R}^+).$$

As in [5] we can show that every Wiener-Hopf operator $T$ has a representation by a convolution. More precisely, there exists a distribution $\mu$ such that

$$Tf = P^+(\mu * f),\ \forall f \in C_c^\infty(\mathbb{R}^+).$$

If $\phi \in C_c(\mathbb{R})$, then the operator

$$L^p_\omega(\mathbb{R}^+) \ni f \longrightarrow P^+(\phi * f)$$

is a Wiener-Hopf operator and we will denote it by $T_\phi$. Moreover, we have

$$(P^+\mathbf{S}_{-a}\mathbf{S}_a)f = f,\ \forall f \in L^p_\omega(\mathbb{R}^+),$$

but it is obvious that

$$(\mathbf{S}_a P^+ \mathbf{S}_{-a})f \neq f,$$

for all $f \in L^p_\omega(\mathbb{R}^+)$ with a support not included in $]a, +\infty[$. The fact that $\mathbf{S}_a$ is not invertible leads to many difficulties in contrast to the case when we deal with the space $L^p_\omega(\mathbb{R})$.

Let $\mathbf{E}$ be the space $L^p_\omega(\mathbb{R}^+)$. As above define

$$\mathfrak{a}_0 = \lim_{t\to+\infty} \ln \|\mathbf{S}_t\|^{\frac{1}{t}},\ \mathfrak{a}_1 = \lim_{t\to+\infty} \ln \|\mathbf{S}_{-t}\|^{\frac{1}{t}}$$

and set $J = [-\mathfrak{a}_1, \mathfrak{a}_0]$. The next theorem is similar to Theorem 1.

**Theorem 3.** *Let $a \in J$ and let $T$ be a Wiener-Hopf operator. Then for every $f \in L^p_\omega(\mathbb{R}^+)$ such that $(f)_a \in L^2(\mathbb{R}^+)$, we have*

$$(Tf)_a = P^+\mathcal{F}^{-1}(h_a\widehat{(f)_a}) \tag{1.4}$$

*with $h_a \in L^\infty(\mathbb{R})$ and*

$$\|h_a\|_\infty \leq C\|T\|,$$

*where $C$ is a constant independent of $a$. Moreover, if $\mathfrak{a}_1 + \mathfrak{a}_0 > 0$, the function $h$ defined on $\mathcal{U} = \{z \in \mathbb{C} :\ \operatorname{Im} z \in J\}$ by $h(z) = h_{\operatorname{Im} z}(\operatorname{Re} z)$ is holomorphic on $\overset{\circ}{\mathcal{U}}$.*

**Definition 2.** *The function $h$ defined in Theorem 3 is called the symbol of $T$.*

We are able to examine the spectrum of the operators in the space $\mathcal{W}$ of bounded operators on $\mathbf{E}$ commuting with $(\mathbf{S}_t)_{t\geq 0}$ or $(P^+\mathbf{S}_{-t})_{t\geq 0}$.

Let $\mathcal{O} = \{z \in \mathbb{C},\ \operatorname{Im} z < \mathfrak{a}_0\}$ and $\mathcal{V} = \{z \in \mathbb{C},\ \operatorname{Im} z < \mathfrak{a}_1\}$.

**Theorem 4.** *We have*

$$i)\ \sigma(\mathbf{S}_t) = \{z \in \mathbb{C},\ |z| \leq e^{\mathfrak{a}_0 t}\},\ \forall t > 0. \tag{1.5}$$

$$ii)\ \sigma(P^+\mathbf{S}_{-t}) = \{z \in \mathbb{C},\ |z| \leq e^{\mathfrak{a}_1 t}\},\ \forall t > 0. \tag{1.6}$$

*Let $T \in \mathcal{W}$ and let $\mu_T$ be the symbol of $T$.*
*iii) If $T$ commutes with $\mathbf{S}_t$, $\forall t \geq 0$, then we have*

$$\overline{\mu_T(\mathcal{O})} \subset \sigma(T). \tag{1.7}$$

*iv) If $T$ commutes with $P^+\mathbf{S}_{-t}$, $\forall t \geq 0$, then we have*

$$\overline{\mu_T(\mathcal{V})} \subset \sigma(T). \tag{1.8}$$

The equalities (1.5),(1.6) generalize the well known results for the spectra of the right and left shifts in the space of sequences $l^2$ (see for instance, [10]). However, our proofs are based heavily on the existence of symbols for Wiener-Hopf operators and having in mind Theorem 3, we follow the arguments in [9].

In section 4, we obtain a sharp spectral result for Wiener-Hopf operators having the form $T_\phi$ with $\phi \in C_c(\mathbb{R})$. This result is established here for operators in spaces $L^p_\omega(\mathbb{R}^+)$. It is important to note that even for $p = 2$ and for the Hilbert space $L^2_\omega(\mathbb{R}^+)$ our result below is new.

**Theorem 5.** *Let $\phi \in C_c(\mathbb{R})$. Then*
*i) if supp $(\phi) \subset \mathbb{R}^+$, we have*

$$\overline{\hat{\phi}(\mathcal{O})} = \sigma(T_\phi).$$

*ii) if supp $(\phi) \subset \mathbb{R}^-$, we have*

$$\overline{\hat{\phi}(\mathcal{V})} = \sigma(T_\phi).$$

The above result yields a weak spectral mapping property and can be compared with the equality (1.3) in Theorem 2, however the proof is more complicated.

## 2. Multipliers on $L^p_\omega(\mathbb{R})$

Recall that we use the notation $E = L^p_\omega(\mathbb{R})$. We start with the following

**Lemma 1.** *Let $\lambda \in \mathbb{C}$ be such that $e^\lambda \in \sigma(S)$ and let $\operatorname{Re}\lambda = \alpha_0$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions of $E$ and an integer $k \in \mathbb{Z}$ so that*

$$\lim_{n\to\infty} \left\| \left(e^{tA} - e^{(\lambda + 2\pi k i)t}\right) f_n \right\| = 0,\ \forall t \in \mathbb{R},\ \ \|f_n\| = 1,\ \forall n \in \mathbb{N}. \tag{2.1}$$

**Proof.** Let $A$ be the generator of the group $(S_t)_{t\in\mathbb{R}}$. It is clear that the group $(S_t)_{t\in\mathbb{R}}$ preserves positive functions. Since $E = L^p_\omega(\mathbb{R})$ the results of [12], [13] say that the spectral mapping theorem holds and

$$\sigma(e^{tA}) \setminus \{0\} = e^{t\sigma(A)} = \{e^{t\lambda} : \lambda \in \sigma(A)\}.$$

In particular, for the spectral bound $s(A)$ of $A$ we get

$$s(A) := \sup\{\operatorname{Re} z : z \in \sigma(A)\} = \alpha_0.$$

Thus $e^\lambda \in \sigma(S) \setminus \{0\} = e^{\sigma(A)}$ yields $\lambda + 2\pi ki = \lambda_0 \in \sigma(A)$ for some $k \in \mathbb{Z}$. On the other hand, $\operatorname{Re}\lambda_0 = \alpha_0$, and we deduce that $\lambda_0$ is on the boundary of the spectrum of $A$. By a well known result, this implies that $\lambda_0$ is in the approximative point spectrum of $A$.

Let $\mu_n$ be a sequence such that $\mu_n \to_{n\to\infty} \lambda_0$, $\operatorname{Re}\mu_n > \lambda_0$, $\forall n \in \mathbb{N}$. Then

$$\left\|(\mu_n I - A)^{-1}\right\| \geq (\text{dist}\,(\mu_n, \sigma(A)))^{-1},$$

hence $\|(\mu_n I - A)^{-1}\| \to \infty$. Applying the uniform boundedness principle and passing to a subsequence of $\mu_n$ (for simplicity also denoted by $\mu_n$), we may find $f \in E$ such that

$$\lim_{n\to\infty} \left\|(\mu_n I - A)^{-1} f\right\| \to \infty.$$

Introduce $f_n \in D(A)$ defined by

$$f_n = \frac{(\mu_n I - A)^{-1} f}{\|(\mu_n I - A)^{-1} f\|}.$$

The identity

$$(\lambda + 2\pi ki - A) f_n = (\lambda_0 - \mu_n) f_n + (\mu_n - A) f_n$$

implies that $(\lambda + 2\pi ki - A) f_n \to 0$ as $n \to \infty$. Then the equality

$$(e^{tA} - e^{t(\lambda+2\pi ki)}) f_n = \Big(\int_0^t e^{(\lambda+2\pi ki)(t-s)} e^{As} ds\Big)(A - \lambda - 2\pi ki) f_n$$

yields (2.1). $\square$

Now we prove the following important lemma.

**Lemma 2.** *For all $\phi \in C_c^\infty(\mathbb{R})$ and $\lambda$ such that $e^\lambda \in \sigma(S)$ with $\operatorname{Re}\lambda = \alpha_0$ we have*

$$|\hat{\phi}(i\lambda + a)| \leq \|M_\phi\|, \ \forall a \in \mathbb{R}. \tag{2.2}$$

**Proof.** Let $\lambda \in \mathbb{C}$ be such that $e^\lambda \in \sigma(S)$ and $\operatorname{Re}\lambda = \alpha_0$ and let $(f_n)_{n\in\mathbb{N}}$ be the sequence constructed in Lemma 1. We have

$$1 = \|f_n\| = \sup_{g\in E',\, \|g\|_{E'}\leq 1} | < f_n, g > |.$$

Then, there exists $g_n \in E'$ such that

$$| < f_n, g_n > -1| \leq \frac{1}{n}$$

and $\|g_n\|_{E'} \leq 1$. Fix $\phi \in C_c^\infty(\mathbb{R})$ and consider

$$|\hat{\phi}(i\lambda + a)| \leq |\hat{\phi}(i\lambda + a)\langle f_n, g_n\rangle| + \frac{1}{n}|\hat{\phi}(i\lambda + a)|$$

$$\leq \Big|\int_{\mathbb{R}} \langle \phi(t)\Big(e^{(\lambda+2\pi ki)t} - S_t\Big)e^{-i(a+2\pi k)t} f_n, g_n\rangle dt\Big| + \frac{1}{n}|\hat{\phi}(i\lambda a)|$$

$$+\Big|\int_{\mathbb{R}} \langle \phi(t) S_t e^{-i(a+2\pi k)t} f_n, g_n\rangle dt\Big|.$$

The first two terms on the right side of the last inequality go to 0 as $n \to \infty$ since by Lemma 1 we have

$$\Big\|e^{-i(a+2\pi k)t}\Big(e^{(\lambda+2\pi ki)t} - S_t\Big) f_n\Big\| \longrightarrow_{n\to\infty} 0.$$

On the other hand,

$$I_n = \Big|\int_{\mathbb{R}} < \phi(t) S_t e^{-i(a+2\pi k)t} f_n, g_n > dt\Big|$$

$$= \Big|\langle \Big[\int_{\mathbb{R}} \phi(t) e^{-i(a+2\pi k)t} f_n(.-t)dt\Big], g_n\rangle\Big|$$

$$= \Big|\langle \int_{\mathbb{R}} (\phi(.-y) e^{i(a+2\pi k)y} f_n(y)dy, e^{i(a+2\pi k).} g_n\rangle\Big|$$

$$= \Big|\langle \Big(M_\phi(e^{i(a+2\pi k).} f_n)\Big), e^{i(a+2\pi k).} g_n\rangle\Big|$$

and $I_n \leq \|M_\phi\| \|f_n\| \|g_n\|_{E'} \leq \|M_\phi\|$. Consequently, we deduce that

$$|\hat{\phi}(i\lambda + a)| \leq \|M_\phi\|.$$

□

Notice that the property (2.2) implies that

$$|\hat{\phi}(\lambda)| \leq \|M_\phi\|, \ \forall \lambda \in \mathbb{C}, \text{ provided } \operatorname{Im} \lambda = \alpha_0.$$

**Lemma 3.** *Let $\phi \in C_c^\infty(\mathbb{R})$ and let $\lambda$ be such that $e^{-\bar{\lambda}} \in \sigma((S_{-1})^*)$ with $Re\, \lambda = -\alpha_1$. Then we have*

$$|\hat{\phi}(i\lambda + a)| \leq \|(M_\phi)\|, \ \forall a \in \mathbb{R}. \tag{2.3}$$

**Proof.** Consider the group $(S_{-t})^*_{t\in\mathbb{R}}$ acting on $E'$. Let $\lambda \in \mathbb{C}$ be such that $e^{-\bar{\lambda}} \in \sigma((S_{-1})^*)$ and

$$|e^{-\bar{\lambda}}| = \rho(S_{-1}) = \rho((S_{-1})^*) = e^{\alpha_1}.$$

The group $(S_{-t})^*$ preserves positive functions. To prove this, assume that $g(x) \geq 0$, *a.e.* is a positive function and let $h \in E$ be such that $h(x) \geq 0$, *a.e.* Then

$$\langle h, (S_{-t})^* g\rangle = \langle S_{-t} h, g\rangle \geq 0.$$

If $F(x) = ((S_{-t})^* g)(x) < 0$ for $x \in \Lambda \subset \mathbb{R}$ and $\Lambda$ has a positive measure, we choose $h(x) = \mathbf{1}_\Lambda(x)$. Then $\langle \mathbf{1}_\Lambda, F\rangle \leq 0$ and we conclude that $F(x) = 0$ a.e. in $\Lambda$ which is a contradiction. For the group $(S_{-t})^*$ the spectral mapping theorem holds and, by the

same argument as in Lemma 1, we prove that there exists a sequence $(g_k)_{k\in\mathbb{N}}$ of functions of $E'$ and an integer $m$ so that for all $t \in \mathbb{R}$,

$$\lim_{k\to\infty} \|(e^{tB} - e^{(-\bar{\lambda}+2\pi mi)t)})g_k\|_{E'} = 0$$

and $\|g_k\|_{E'} = 1$.

Since $S_{-t}S_t = I$, we have $(S_t)^*(S_{-t})^* = I$. This implies that

$$\big\|(S_t)^*g_k - e^{(\bar{\lambda}-2\pi mi)t}g_k\big\|_{E'} = \Big\|\Big((S_t)^* - e^{(\bar{\lambda}-2\pi mi)t}(S_t)^*(S_{-t})^*\Big)g_k\Big\|_{E'}$$

$$\leq \|(S_t)^*\|_{E'\to E'}\Big\|\Big(e^{(-\bar{\lambda}+2\pi mi)t} - (S_{-t})^*\Big)g_k\Big\|_{E'}$$

and we deduce that for every $t \in \mathbb{R}$ we have

$$\lim_{k\to\infty} \Big\|\Big((S_t)^* - e^{(\bar{\lambda}-2\pi mi)t}\Big)g_k\Big\|_{E'} = 0.$$

For $1 < p < +\infty$ the space $E = L^p_\omega(\mathbb{R})$ is reflexive and the dual to $E'$ can be identified with $E$. Consequently, since $\|g_k\|_{E'} = 1$, there exists $f_k \in E$ such that

$$| < f_k, g_k > -1| \leq \frac{1}{k},\ \ \|f_k\|_E \leq 1. \tag{2.4}$$

For $p = 1$ the space $L^1_\omega(\mathbb{R})$ is not reflexive and to arrange (2.4), we use another argument. In this case the dual to $L^1_\omega(\mathbb{R})$ is $L^\infty_\omega(\mathbb{R})$. Let $\|g\|_{L^\infty_\omega(\mathbb{R})} = 1$. Fix $0 < \epsilon < 1$ and consider the set

$$\mathcal{M}_{\epsilon,m} = \{x \in \mathbb{R} :\ |g(x)|\omega(x) \geq 1-\epsilon,\ m \leq x < m+1\},\ m \in \mathbb{Z}.$$

If $\mu(\mathcal{M}_{\epsilon,m})$ (the Lebesgue measure of $\mathcal{M}_{\epsilon,m}$) is zero for all $m \in \mathbb{Z}$, we obtain a contradiction with $\|g\|_{L^\infty_\omega(\mathbb{R})} = 1$. Thus there exists $r \in \mathbb{Z}$ such that $\mu(\mathcal{M}_{\epsilon,r}) > 0$. Now we take

$$f(x) = \frac{\mathbf{1}_{\mathcal{M}_{\epsilon,r}}(x)e^{i\arg(g(x))}}{\mu(\mathcal{M}_{\epsilon,r})\omega^2(x)}.$$

Then

$$1 \geq \langle f, g\rangle = \int_r^{r+1} f(x)\bar{g}(x)\omega^2(x)dx \geq 1-\epsilon$$

and we can obtain (2.4) choosing $\epsilon = 1/k$. Passing to the proof of (2.3), we get

$$|\hat{\phi}(i\lambda + a)| \leq |\hat{\phi}(i\lambda + a) < f_k, g_k > | + \frac{1}{k}|\hat{\phi}(i\lambda + a)|$$

$$\leq \Big|\int_{\mathbb{R}} < \phi(t)e^{-i(a+2\pi m)t}f_k, \Big(e^{(\bar{\lambda}-2\pi mi)t} - (S_t)^*\Big)g_k > dt\Big|$$

$$+\Big|\int_{\mathbb{R}} < \phi(t)S_t\Big(e^{-i(a+2\pi m)t}f_k\Big), g_k > dt\Big| + \frac{1}{k}|\hat{\phi}(i\lambda + a)| = J'_k + I'_k + \frac{1}{k}|\hat{\phi}(i\lambda + a)|.$$

From the argument above we deduce that $J'_k \to 0$ as $k \to \infty$. For $I'_k$ we apply the same argument as in the proof of Lemma 2 and we deduce

$$|\hat{\phi}(i\lambda + a)| \leq \|M_\phi\|. \ \Box$$

For the proof of Theorem 1 we apply the argument in [7] and Lemmas 2-3. There exists $e^{\lambda_0} \in \sigma(S)$ such that $\operatorname{Re}\lambda_0 = \alpha_0$. Then for every $z \in \mathbb{C}$ with $\operatorname{Im} z = \alpha_0$ we have

$$|\hat{\varphi}(z)| \leq \|M_\varphi\|.$$

Also there exists $e^{-\lambda_1} \in \sigma((S_{-1})^*)$ with $\operatorname{Re}\lambda_1 = -\alpha_1$ and for every $z \in \mathbb{C}$ with $\operatorname{Im} z = -\alpha_1$ we have

$$|\hat{\varphi}(z)| \leq \|M_\varphi\|.$$

Applying Phragmen-Lindelöff theorem for the Fourier transform of $\varphi \in C_c^\infty(\mathbb{R})$ in the domain $\{z \in \mathbb{C} : \, -\alpha_1 \leq \operatorname{Im} z \leq \alpha_0\}$, we deduce

$$|\hat{\varphi}(z)| \leq \|M_\varphi\|$$

for $z \in U$. Next we exploit the fact that $M$ can be approximated by $M_\varphi$ with respect to the strong operator topology (see [6] for a very general setup covering our case). We complete the proof repeating the arguments from [6], [7] and since this leads to minor modifications, we omit the details. To obtain Theorem 2 we follow the same argument as in [8] and the proof is omitted.

To see that in (1.2) the inclusion may be strict, consider a measure $\eta$ on $\mathbb{R}$ such that the operator

$$M_\eta : f \longrightarrow \int_{\mathbb{R}} S_x(f) d\eta(x)$$

is bounded on $L^1(\mathbb{R})$. For this it is enough to have $\int_{\mathbb{R}} d|\eta|(x) < \infty$. Then $M_\eta$ is a multiplier on $L^1(\mathbb{R})$ with symbol

$$\hat{\eta}(t) = \int_{\mathbb{R}} e^{-ixt} d\eta(x).$$

On the other hand, there exists a bounded measure $\eta$ on $\mathbb{R}$ such that

$$\overline{\hat{\eta}(\mathbb{R})} \neq \sigma(M_\eta)$$

(see for details [2]). In $L^1(\mathbb{R})$ we have $\alpha_0 = \alpha_1 = 0$ and $U = \mathbb{R}$. So we have not the property (1.2) in Theorem 2 for every multiplier even in the case $L^1(\mathbb{R})$.

## 3. Wiener-Hopf operators

We need the following lemmas.

**Lemma 4.** *Let $\phi \in C_c(\mathbb{R}^+)$. The operator $T_\phi$ commutes with $\mathbf{S}_t$, $\forall t > 0$, if and only if the support of $\phi$ is in $\overline{\mathbb{R}^+}$.*

**Proof.** Consider $\phi \in C_c(\mathbb{R}^+)$ and suppose that $T_\phi$ commutes with $\mathbf{S}_t$, $t \geq 0$. We write

$$\phi = \phi\chi_{\mathbb{R}^-} + \phi\chi_{\mathbb{R}^+}.$$

If $T_\phi$ commutes with $\mathbf{S}_t$, $t \geq 0$, then the operator $T_{\phi\chi_{\mathbb{R}^-}}$ commutes too. Let $\psi = \phi\chi_{\mathbb{R}^-}$ and fix $a > 0$ such that $\psi$ has a support in $[-a, 0]$. Setting $f = \chi_{[0,a]}$, we get $\mathbf{S}_a f = \chi_{[a,2a]}$. For $x \geq 0$ we have

$$P^+(\psi * \mathbf{S}_a f)(x) = \int_{-a}^{0} \psi(t)\chi_{\{a \leq x-t \leq 2a\}} dt = \int_{max(-a,-2a+x)}^{min(x-a,0)} \psi(t)dt.$$

Since $P^+(\psi * \mathbf{S}_a f) = \mathbf{S}_a P^+(\psi * f)$, for $x \in [0, a]$, we deduce $P^+(\psi * \mathbf{S}_a f)(x) = 0$ and

$$\int_{-a}^{x-a} \psi(t)dt = 0,\ \forall x \in [0, a].$$

This implies that $\psi(t) = 0$, for $t \in [-a, 0]$ hence $supp(\phi) \subset \overline{\mathbb{R}^+}$.

□

Next we establish the following

**Lemma 5.** *Let $T_\phi$, $\phi \in C_c(\mathbb{R})$. Then $T_\phi$ commutes with $P^+(\mathbf{S}_{-t})$, $\forall t > 0$ if and only if $supp(\phi) \subset \overline{\mathbb{R}^-}$.*

**Proof.** For $\phi \in C_c(\mathbb{R})$, suppose that $T_\phi$ commutes with $P^+(\mathbf{S}_{-t})$, $\forall t > 0$. Set $\psi = \phi\chi_{\mathbb{R}^+}$. There exists $a > 0$ such that $supp(\psi) \subset [0, a]$. We have $P^+(\psi * P^+\mathbf{S}_{-a}\chi_{[0,a]}) = 0$ and then $P^+\mathbf{S}_{-a}(P^+\psi * \chi_{[0,a]}) = 0$. This implies that

$$(\psi * \chi_{[0,a]})(x) = 0,\ \forall x > a.$$

On the other hand, for $x > a$ we have

$$(\psi * \chi_{[0,a]})(x) = \int_{\mathbb{R}} \psi(t)\chi_{[0,a]}(x-t)dt = \int_{max(0,x-a)}^{min(a,x)} \psi(t)dt = \int_{x-a}^{a} \psi(t)dt.$$

Hence $\int_\epsilon^a \psi(t)dt = 0, \forall a > \epsilon > 0$ and we get $\psi = 0$. Thus we conclude that $supp(\phi) \subset \overline{\mathbb{R}^-}$.

□

It is clear that $(\mathbf{S}_t)_{t\geq 0}$ and $(P^+(\mathbf{S}_{-t}))_{t\geq 0}$ form continuous semigroups and these semigroups preserve positive functions. Moreover, by using the equality

$$\langle (P^+\mathbf{S}_t)h, g\rangle = \langle h, (P^+\mathbf{S}_{-t})^* g\rangle,$$

we conclude that the semigroup $(P^+\mathbf{S}_{-t})^*$ preserve positive functions. The issue is that for $\mathbf{S}_t$ and $(P^+\mathbf{S}_{-t})^*$ the spectral mapping theorem holds and we may repeat the arguments used in section 2. Thus we obtain the following

**Lemma 6.** 1) *For all $\phi \in C_c^\infty(\mathbb{R})$ such that $supp(\phi) \subset \mathbb{R}^+$, for $\lambda$ such that $e^\lambda \in \sigma(\mathbf{S})$ and* $\operatorname{Re}\lambda = \mathfrak{a}_0$, *we have*

$$|\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|,\ \forall a \in \mathbb{R}.$$

2) *For all $\phi \in C_c^\infty(\mathbb{R})$ such that $supp(\phi) \subset \mathbb{R}^-$ and for $\lambda$ such that $e^{-\bar{\lambda}} \in \sigma((\mathbf{S}_{-1})^*)$ and $Re\,\lambda = -\mathfrak{a}_1$, we have*

$$|\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|,\ \forall a \in \mathbb{R}.$$

**Proof.** Let $A$ be the generator of the semi-group $(\mathbf{S}_t)_{t\geq 0}$. First we obtain using the same arguments as in the proof of Lemma 1 that for $\lambda$ such that $e^{\lambda} \in \sigma(\mathbf{S})$ and $\operatorname{Re}\lambda = \mathfrak{a}_0$, there exists a sequence $(f_n)$ of functions of $\mathbf{E}$ and an integer $k \in \mathbb{Z}$ so that

$$\lim_{n\to\infty} \left\| \left(e^{tA} - e^{(\lambda+2k\pi i)t}\right) f_n \right\| = 0,\ \forall t \in \mathbb{R}^+,\ \ \|f_n\| = 1,\ \forall n \in \mathbb{N}.$$

Then we notice that

$$\begin{aligned}\|(P^+\mathbf{S}_{-t} - e^{-(\lambda+2k\pi i)t})f_n\| &= \|(P^+\mathbf{S}_{-t} - e^{-(\lambda+2k\pi i)t}P^+\mathbf{S}_{-t}\mathbf{S}_t)f_n\| \\ &\leq \|P^+\mathbf{S}_{-t}\||e^{-(\lambda+2k\pi i)t}|\|(e^{(\lambda+2k\pi i)t} - \mathbf{S}_t)f_n\|,\ \forall t \in \mathbb{R}^+.\end{aligned}$$

Thus

$$\lim_{n\to+\infty} \|(P^+\mathbf{S}_{-t} - e^{-(\lambda+2k\pi i)t})f_n\| = 0,\ \forall t \in \mathbb{R}^+.$$

So we have

$$\lim_{n\to+\infty} \left\| \left(P^+\mathbf{S}_t - e^{(\lambda+2k\pi i)t}\right) f_n \right\| = 0,\ \forall t \in \mathbb{R}.$$

Using the same arguments as in the proof of Lemma 2, we obtain

$$|\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|,\ \forall a \in \mathbb{R},\ \forall \phi \in C_c^\infty(\mathbb{R})$$

and $\lambda$ such that $e^{\lambda} \in \sigma(\mathbf{S})$ and $\operatorname{Re}\lambda = \mathfrak{a}_o$. In the same way we prove 2) using the semi-group $((P^+\mathbf{S}_{-t})^*)_{t\geq 0}$. $\square$

To establish Theorem 3, we use Lemma 6 and we follow with trivial modifications the arguments in [5], [7], [8]. We omit the details. For the proof of Theorem 4 we repeat the arguments in [9].

Now we pass to the proof of Theorem 5.

**Proof of Theorem 5.**

Let $\mathcal{A}$ be the commutative algebra generated by $T_\phi$ for all $\phi$ in $C_c(\mathbb{R}^+)$ with support in $\mathbb{R}^+$ and $\mathbf{S}_x$, for all $x \in \mathbb{R}^+$.

Denote by $\widehat{\mathcal{A}}$ the set of the characters on $\mathcal{A}$. Let $\beta \in \sigma(T_\phi) \setminus \{0\}$. Then there exists $\gamma \in \widehat{\mathcal{A}}$ such that $\beta = \gamma(T_\phi)$. We will prove the following equality

$$\gamma(T_\phi) = \int_{\mathbb{R}^+} \phi(x)\gamma(S_x)dx.$$

This result il not trivial because we cannot commute $\gamma$ with the Bochner integral $\int_{\mathbb{R}^+} \phi(x)S_x dx$.

Set

$$\theta_\gamma(x) = \gamma(\mathbf{S}_x) = \frac{\gamma(\mathbf{S}_x \circ T_\phi)}{\gamma(T_\phi)},\ \forall x \in \mathbb{R}^+.$$

Let $\psi \in C_c(\mathbb{R}^+)$ and let $supp(\psi) \subset K$, where $K$ is a compact subset of $\mathbb{R}^+$.

Suppose that $(\psi_n)_{n\geq 0} \subset C_K(\mathbb{R}^+)$ is a sequence converging to $\psi$ uniformly on $K$.

For every $g \in \mathbf{E}$, we get

$$\|T_{\psi_n}g - T_\psi g\| \leq \|\psi_n - \psi\|_\infty \sup_{y\in K} \|\mathbf{S}_y\|\|g\|$$

and this implies that $\lim_{n\to+\infty} \|T_{\psi_n} - T_\psi\| = 0$. This shows that the linear map $\psi \longrightarrow T_\psi$ is sequentially continuous and hence it is continuous from $C_c(\mathbb{R}^+)$ into $\mathcal{A}$. Since the map

$$x \longrightarrow \mathbf{S}_x(\psi)$$

is continuous from $\mathbb{R}^+$ into $C_c(\mathbb{R}^+)$, we conclude that the map

$$x \longrightarrow \mathbf{S}_x \circ T_\phi = T_{\mathbf{S}_x(\phi)}$$

is continuous from $\mathbb{R}^+$ into $\mathcal{A}$. Consequently, the function $\theta_\gamma$ is continuous on $\mathbb{R}^+$. Introduce

$$\eta : C_c(\mathbb{R}^+) \ni \psi \longrightarrow \gamma(T_\psi).$$

The map $\eta$ is a continuous linear form on $C_c(\mathbb{R}^+)$ and applying Riesz representation theorem, there exists some Borel measure $\mu$ (see for instance, [11]) such that

$$\eta(\psi) = \int_{\mathbb{R}^+} \psi(x) d\mu(x), \ \forall \psi \in C_c(\mathbb{R}^+).$$

This implies that for all $f$, $\psi \in C_c(\mathbb{R}^+)$, we have

$$\gamma(T_\psi \circ T_f) = \int_{\mathbb{R}^+} (\psi * f)(t) d\mu(t)$$

$$= \int_{\mathbb{R}^+} \Big( \int_{\mathbb{R}^+} \psi(x) f(t-x) dx \Big) d\mu(t).$$

Using the Fubini theorem, we obtain

$$\gamma(T_\psi \circ T_f) = \int_{\mathbb{R}^+} \psi(x) \Big( \int_{\mathbb{R}^+} f(t-x) d\mu(t) \Big) dx = \int_{\mathbb{R}^+} \psi(x) \gamma(\mathbf{S}_x \circ T_f) dx$$

and replacing $f$ and $\psi$ by $\phi$, we get

$$\gamma(T_\phi) = \int_{\mathbb{R}^+} \phi(x) \theta_\gamma(x) dx, \ \forall \phi \in C_c(\mathbb{R}^+). \tag{3.1}$$

Notice that $\theta_\gamma(x+y) = \theta_\gamma(x)\theta_\gamma(y)$, $\forall x, y \in \mathbb{R}^+$ . We will prove that $\theta_\gamma(x) \neq 0$, $\forall x \in \mathbb{R}^+$. Suppose $\theta_\gamma(x_0) = 0$, for $x_0 > 0$. Then $\gamma(\mathbf{S}_{x_0}) = \Big(\gamma(\mathbf{S}_{\frac{x_0}{n}})\Big)^n = 0$ and $\theta_\gamma(\frac{x_0}{n}) = \gamma\Big(\mathbf{S}_{\frac{x_0}{n}}\Big) = 0$ for every $n \in \mathbb{N}$. Since $\theta_\gamma$ is continuous on $\mathbb{R}^+$,

$$\lim_{n\to+\infty} \theta_\gamma\Big(\frac{x_0}{n}\Big) = \theta_\gamma(0) = 1$$

and we obtain a contradiction. Consequently, we have $\theta_\gamma(x) = \gamma(\mathbf{S}_x) \neq 0$, for all $x \in \mathbb{R}^+$. Now define $\theta_\gamma(-x) = \frac{1}{\theta_\gamma(x)}$, $\forall x \in \mathbb{R}^+$. It is easy to check that $\theta_\gamma$ is a morphism on $\mathbb{R}$. It is clear that $\theta_\gamma(x+y) = \theta_\gamma(x)\theta_\gamma(y)$, for $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ and for $(x, y) \in \mathbb{R}^- \times \mathbb{R}^-$. Suppose that $x > y > 0$,

$$\theta_\gamma(x-y) = \gamma(\mathbf{S}_x\mathbf{S}_{-y}) = \frac{\gamma(\mathbf{S}_x\mathbf{S}_{-y}\mathbf{S}_y)}{\gamma(\mathbf{S}_y)} = \frac{\theta_\gamma(x)}{\theta_\gamma(y)} = \theta_\gamma(x)\theta_\gamma(-y).$$

Moreover,

$$\theta_\gamma(y-x) = \frac{1}{\theta_\gamma(x-y)} = \frac{1}{\theta_\gamma(x)\theta_\gamma(-y)} = \theta_\gamma(y)\theta_\gamma(-x).$$

Since $\theta_\gamma$ satisfies $\theta_\gamma(x+y) = \theta_\gamma(x)\theta_\gamma(y)$, for all $(x,y) \in \mathbb{R}^2$, it is well known that this implies that there exists $\lambda \in \mathbb{C}$ such that $\theta_\gamma(x) = e^{\lambda x}$, for all $x \in \mathbb{R}$.

On the other hand, we have $\gamma(\mathbf{S}_x) \in \sigma(\mathbf{S}_x)$ and $\gamma(\mathbf{S}_1) = e^\lambda \in \sigma(\mathbf{S})$. Thus (3.1) implies

$$\beta = \gamma(T_\phi) = \hat{\phi}(-i\lambda)$$

with $\lambda \in \mathcal{O}$. We conclude that

$$\sigma(T_\phi) \setminus \{0\} \subset \hat{\phi}(\mathcal{O}).$$

Now, suppose that $supp(\phi) \subset \mathbb{R}^-$. Let $\mathcal{B}$ be the commutative Banach algebra generated by $T_\psi$ for all $\psi \in C_c(\mathbb{R}^-)$ and by $P^+\mathbf{S}_{-x}$, for all $x \in \mathbb{R}^+$. Let $\kappa \in \sigma(T_\phi)$. Using the same arguments as above, and the set of characters $\widehat{\mathcal{B}}$ of $\mathcal{B}$, we get

$$\kappa = \int_{\mathbb{R}^-} \phi(x) e^{\delta x} dx,$$

with $-i\delta \in \mathcal{V}$. This completes the proof of Theorem 5.

□

## 4. Comments and open problems

Following the general schema of the proof of the existence of symbols for multipliers developed in [6] for locally compact abelian groups, it is natural to conjecture that an analog of Theorem 1 holds for general Banach spaces of functions under some hypothesis as we have proved this for general Hilbert space of functions in [7], [9]. Using the notations of Section 2, the crucial point is the inequality

$$|\hat{\varphi}(z)| \leq \|M_\varphi\|, \ \forall \varphi \in C_c^\infty(\mathbb{R}), \ \operatorname{Im} z = \alpha_0. \tag{4.1}$$

and a similar inequality for $\operatorname{Im} z = -\alpha_1$. To establish (4.1), we introduced the factor $\langle f_k, g_k \rangle$ (see proof of Lemma 1) close to 1 and we want to estimate $\hat{\varphi}(z)\langle f_k, g_k\rangle$. Here the sequence $f_k, \|f_k\| = 1$, must be chosen so that for some integers $n_k \in \mathbb{Z}$ and $e^\lambda \in \sigma(S)$, $\operatorname{Re} \lambda = \alpha_0$, we have

$$\lim_{k \to \infty} \|(S_t - e^{(\lambda + 2\pi n_k i)t}) f_k\| = 0, \ \forall t \in \mathbb{R}. \tag{4.2}$$

If the spectral mapping theorem is true for the group $S_t = e^{At}$, we have $s(A) = \alpha_0$ and (4.2) can be obtained as in Section 2. On the other hand, if $s(A) < \alpha_0$, we may construct $(f_k)$ assuming that

$$\sup_{m \in \mathbb{R}} \|(A - \alpha_0 - 2\pi m i)^{-1}\| = +\infty. \tag{4.3}$$

For Hilbert spaces (4.3) holds (see [3], [4], [1]) and author has exploited this property in [8], [7] to complete the proof of (4.2). For semigroups in Banach spaces $s(A) < \alpha_0$ does not implies in general (4.3)

(see a counter-example in Chapter V in [1] and the relation between the resolvent of $A$ and the spectrum of $S_t$ in [4]). Consequently, it is not possible to use (4.3) and to construct a sequence $f_k$ for which (4.2) holds. Of course another proof of (4.1) could be possible, and in Banach spaces of functions for which $s(A) < \alpha_0$ this is an open problem.

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LMAM, Université de Metz, UMR 7122,Ile du Saulcy 57045, Metz Cedex 1, France.
*E-mail address*: `petkova@univ-metz.fr`